

# ON THE DYNAMICAL SOLUTION OF QUANTUM MEASUREMENT PROBLEM

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**ABSTRACT.** The development of quantum measurement theory, initiated by von Neumann, only indicated a possibility for resolution of the interpretational crisis of quantum mechanics. We do this by divorcing the algebra of the dynamical generators and the algebra of the actual observables, or *beables*. It is shown that within this approach quantum causality can be rehabilitated in the form of a superselection rule for compatibility of the past beables with the potential future. This rule, together with the self-compatibility of the measurements insuring the consistency of the histories, is called the nondemolition, or causality principle in modern quantum theory. The application of this rule in the form of the dynamical commutation relations leads in particular to the derivation of the von Neumann projection postulate. This gives a quantum stochastic solution, in the form of the dynamical filtering equations, of the notorious measurement problem which was tackled unsuccessfully by many famous physicists starting with Schrödinger and Bohr.

## 1. INTRODUCTION

*How wonderful we have met with a paradox, now we have some hope of making progress* - Niels Bohr.

In this paper we present the main ideas of modern quantum measurement theory and the author's views on the quantum measurement problem which might not coincide with the present scientific consensus that this problem is unsolvable in the standard framework, or at least unsolved [14]. It will be shown that there exists such solution along the line suggested by the great founders of quantum theory Schrödinger, Heisenberg and Bohr. We shall see that von Neumann only partially solved this problem which he studied in his Mathematical Foundation of Quantum Theory [11], and that the direction in which the solution might be found was envisaged by Schrödinger [20] and J Bell [1].

Recent phenomenological theories of continuous reduction, quantum state diffusion and quantum trajectories extended the instantaneous projection postulate to a certain class of continuous-in-time measurements. As we shall see here, there is no need to supplement the usual quantum mechanics with any of such generalized reduction postulate even in the continuous time. They all have been derived from the time continuous unitary evolution for a generalized Dirac type Schrödinger equation with a singular scattering interaction at the boundary of our Hamiltonian model, see the recent review paper [2]. The quantum causality as a new superselection

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rule provides a time continuous nondemolition measurement in the extended system which enables to obtain the quantum state diffusion and quantum trajectories simply by time continuous conditioning called quantum filtering. Our nondemolition causality principle, which was explicitly formulated in [3], admits to select a continuous diffusive classical process in the quantum extended world which satisfies the nondemolition condition with respect to all future of the measured system. And this allows us to obtain the continuous trajectories for quantum state diffusion by simple filtering of quantum noise exactly as it was done in the classical statistical nonlinear filtering and prediction theory. In this way we derived [4, 5] the quantum state diffusion of a Gaussian wave packet as the result of the solution of quantum prediction problem by filtering the quantum white noise in a quantum stochastic Langevin model for the continuous observation. Thus the “primary” for the conventional quantum mechanics stochastic nonlinear irreversible quantum state diffusion appears to be the secondary, as it should be, to the deterministic linear unitary reversible evolution of the extended quantum mechanics containing necessarily infinite number of auxiliary particles. However quantum causality, which defines the arrow of time by selecting what part of the reversible world is related to the classical past and what is related to the quantum future, makes the extended mechanics irreversible in terms of the injective semigroup of the invertible Heisenberg transformations induced by the unitary group evolution for the positive arrow of time. The microscopic information dynamics of this event enhanced quantum mechanics, or Eventum Mechanics, allows the emergence of the decoherence and the increase of entropy in a purely dynamical way without any sort of reservoir averaging.

Summarizing, we can formulate the general principles of the Eventum Mechanics which unifies the classical and quantum mechanics in such a way that there is no contradiction between the unitary evolution of the matter waves and the phenomenological information dynamics such as quantum state diffusion or spontaneous jumps for the events and the trajectories of the particles. This is a conventional, *non-stochastic* but time asymmetric quantum mechanics in an extended Hilbert space, in which the true and hidden observables, or beables are mathematically distinguished from the evolution generators. It can be described by the following features:

- It is a reversible wave mechanics of the continuous unitary group evolutions in an infinite-dimensional Hilbert space
- It has conventional interpretation for the normalized Hilbert space vectors as state-vectors ( probability amplitudes)
- However not all operators, e.g. the dynamical generator (Hamiltonian), are admissible as the potential observables
- Quantum causality is statistical predictability of the quantum states based on the results of the actual measurements
- It implies the choice of time arrow and an initial state which together with past measurement data defines the reality
- The actual observables (beables) must be compatible with any operator representing a potential (future) observable
- The Heisenberg dynamics and others symmetries induced by unitary operators should be algebraically endomorphic
- However these endomorphisms form only a semigroup on the algebra of all observables as they may be irreversible.

Note that the classical Hamiltonian mechanics can be also described in this way by considering only the commutative algebras of the potential observables. Each such observable is compatible with any other and can be considered as an actual observable, or beable. However, the Hamiltonian operator, generating a non-trivial Liouville unitary dynamics in the corresponding Hilbert space, is not an observable, as it doesn't commute with any observable which is not the integral of motion. Nevertheless the corresponding Heisenberg dynamics, described by the induced automorphisms of the commutative algebra, is reversible, and pure states, describing the reality, remain pure, non disturbed by the measurements of its observables. This is also true in the purely quantum mechanical case, in which the Hamiltonian is an observable, as there are no events and nontrivial beables in the conventional quantum mechanics. The only actual observables, which are compatible with any Hermitian operator as a potential observable, are the constants, i.e. proportional to the identity operator, as the only operators, commuting with any such observable. Their measurements do not bring new information and do not disturb the quantum states. However any non-trivial classical-quantum Hamiltonian interactions cannot induce a group of the reversible Heisenberg automorphisms but only a semigroup of irreversible endomorphisms of the decomposable algebra of all potential observables of the composed classical-quantum system. This follows from the simple fact that any automorphism leaves the center of an operator algebra invariant, and thus induces the autonomous noninteracting dynamics on the classical part of the semi-classical system. This is the only reason which is responsible for failure of all earlier desperate attempts to build the reversible, time symmetric Hamiltonian theory of classical-quantum interaction which would give a dynamical solution of the quantum decoherence and measurement problem along the line suggested by von Neumann and Bohr. There is no nontrivial reversible classical-quantum mechanical interaction, but as we have seen, there is a Hamiltonian irreversible interaction within the time asymmetric Eventum Mechanics.

The unitary solution of the described boundary value problem indeed induces endomorphic semi-classical Hamiltonian dynamics, and in fact is underlying in any phenomenological reduction model [2]. Note that although the irreversible Heisenberg endomorphisms of eventum mechanics, induced by the unitary propagators, are injective, and thus are invertible by completely positive maps, and are not mixed, they mix the pure states over the center of the algebra. Such mixed states, which are uniquely represented as the orthogonal mixture over the 'hidden' variables (beables), can be filtered by the measurement of the actual observables, and this transition from the prior state corresponding to the less definite (mixed) reality to the posterior state, corresponding to a more definite (pure) reality by the simple inference is not change the reality. This is an explanation, in the pure dynamical terms of the eventum mechanics, of the emergence of the decoherence and the reductions due to the measurement, which has no explanation in the conventional classical and quantum mechanics.

Our mathematical formulation of the eventum mechanics as the extended quantum mechanics equipped with the quantum causality to allow events and trajectories in the theory, is just as continuous as Schrödinger could have wished. However, it doesn't exclude the jumps which only appear in the singular interaction picture, which are there as a part of the theory, not only of its interpretation. Although Schrödinger himself didn't believe in quantum jumps, he tried several

times, although unsuccessfully, to obtain the continuous reduction from a generalized, relativistic, “true Schrödinger” equation. He envisaged that ‘if one introduces two symmetric systems of waves, which are traveling in opposite directions; one of them presumably has something to do with the known (or supposed to be known) state of the system at a later point in time’ [20], then it would be possible to derive the ‘verdamnte Quantenspringerei’ for the opposite wave as a solution of the future-past boundary value problem. This desire coincides with the “transactional” attempt of interpretation of quantum mechanics suggested in [21] on the basis that the relativistic wave equation yields in the nonrelativistic limit two Schrödinger type equations, one of which is the time reversed version of the usual equation: ‘The state vector  $\psi$  of the quantum mechanical formalism is a real physical wave with spatial extension and it is identical with the initial “offer wave” of the transaction. The particle (photon, electron, etc.) and the collapsed state vector are identical with the completed transaction.’ There was no proof of this conjecture, and now we know that it is not even possible to derive the quantum state diffusions, spontaneous jumps and single reductions from models involving only a finite particle state vectors  $\psi(t)$  satisfying the conventional Schrödinger equation.

Our new approach, based on the exactly solvable boundary value problems for infinite particle states described in this paper, resolves the problem formulated by Schrödinger. And thus it resolves the old problem of interpretation of the quantum theory, together with its infamous paradoxes, in a constructive way by giving exact nontrivial models for allowing the mathematical analysis of quantum observation processes determining the phenomenological coupling constants and the reality underlying these paradoxes. Conceptually it is based upon a new idea of quantum causality called the nondemolition principle [3] which divides the world into the classical past, forming the consistent histories, and the quantum future, the state of which is predictable for each such history.

Here we develop the discrete time approach introduced in [4, 10] for solving the famous Schrödinger’s cat paradox. We shall see that even the most general quantum decoherence and wave packet reduction problem for an instantaneous or even sequential measurements can be solved in a canonical way which corresponds to adding a single initial cat’s state. The discrete time dynamical model used for this solution is in parallel with the quantum stochastic model for continuous in time measurements suggested in [6, 7], see also [8, 9]. These models give the dynamical justification of the projection and other phenomenological decoherence and reduction postulates. They resolve the Schrödinger cat paradoxes of quantum measurement theory in a constructive way, giving exact nontrivial models in the differential form of evolution equations for the statistical analysis of quantum observation processes determining the reality underlying these paradoxes. Conceptually they are based upon a new idea of quantum causality as a superselection rule called the Nondemolition Principle [3] which divides the world into the classical past, forming the consistent histories, and the quantum future, the state of which is predictable for each such history. This new postulate of the modern quantum theory making the solution of quantum measurement possible can not be contradicted by any experiment as we prove that any sequence of usual, “demolition” measurements based on the projection postulate or any other phenomenological measurement theory is statistically equivalent, and in fact can be dynamically realized as a simultaneous

nondemolition measurement in a canonically extended infinite semi-quantum system. The nondemolition models give exactly the same predictions as the orthodox, “demolition” theories, but they do not require the projection or any other postulate replaced by the nondemolition causality principle. We examine also the implications for time reversibility and time arrow which follow from the quantum causality principle.

## 2. GENERALIZED REDUCTION AND ITS DILATION

Von Neumann’s measurement theory postulates the process of decoherence for any wave-function  $\psi(x)$  as an instantaneous transition, or jump  $\psi\psi^\dagger \mapsto \rho$  into the mixture

$$\rho = \sum_y F(y) \psi \psi^\dagger F(y) = \sum_y \psi_y \psi_y^\dagger \Pr\{y\}$$

of the eigen-functions  $F(y)\psi$  of a discrete-spectrum observable  $Y = \sum y F(y)$ . Here  $F(y)$  is a complete orthogonal family of eigen-projectors

$$YF(y) = yF(y), \quad F(y)^2 = F(y) = F(y)^\dagger, \quad \sum F(y) = I$$

for the observable  $Y$  defining the *posterior state vectors*  $\psi_y = F(y)\psi / \|F(y)\psi\|$  for all measurement results  $y$  which have nonzero probabilities  $\Pr\{y\} = \|F(y)\psi\|^2$ . Note that the projections  $\psi_1(y) = F(y)\psi$  are normalized as

$$\sum_y \|\psi_1(y)\|^2 = \sum_y \int |\psi_1(x, y)|^2 d\lambda_x = 1$$

if  $\|\psi\|^2 := \int |\psi(x)|^2 d\lambda_x = 1$  with respect to a given (discrete or continuous) measure  $\lambda$  on  $x$ . According to the Lüdger’s projection postulate [13], the renormalized non-linear versions  $\psi \mapsto \psi_y$  of the linear transformations  $\psi \mapsto \psi_1(y)$  defines the new states after the measurement corresponding to the measurement results  $y$  (with  $\Pr\{y\} \neq 0$ ).

Obviously the projection postulate is only a phenomenological principle which is inconsistent with the Schrödinger’s unitary evolution, and therefore it requires a dynamical justification. There have been innumerable attempts to derive the decoherence and the projection postulate as a sort of approximation corresponding to a limiting procedure in a dynamical model of the measurement-apparatus interaction. While it is in principle possible to obtain the decoherence as the result of averaging with respect to the additional (reservoir) degrees of freedom, any attempt to derive the projection postulate faces the problem of applying it on a higher level. Surely the nonexistence of the solution for a physically well-defined problem simply indicates an incorrectness of its mathematical setting. It was pointed out by Niels Bohr that it is not possible to resolve this problem unless as the reservoir is considered dynamically as quantum but statistically as classical system. Following this old idea we shall formulate the measurement problem as a mathematical problem which has at least one exact solution. This solution might be not most satisfactory for physics, however it gives the idealized dynamical model for any quantum sequential measurement, not only with discrete but also with continuous spectra. Let us therefore describe the generalized instantaneous reduction principle which includes the indirect measurements with continuous data.

The generalized reduction of the wave function  $\psi(x)$ , corresponding to a complete measurement with discrete or continuous spectrum of  $y$ , is described by a function  $V(y)$  whose values are linear operators  $\mathfrak{h} \ni \psi \mapsto V(y)\psi$  for each  $y$  which are not assumed to be unitary on the quantum system Hilbert space  $\mathfrak{h}$ ,  $V(y)^\dagger V(y) \neq I$ , but have the following normalization condition. The resulting wave-function

$$\psi_1(x, y) = [V(y)\psi](x)$$

is normalized with respect to a given measure  $\mu$  on  $y$  in the sense

$$\iint |[V(y)\psi](x)|^2 d\lambda_x d\mu_y = \int |\psi(x)|^2 d\lambda_x$$

for any probability amplitude  $\psi$  normalized with respect to a measure  $\lambda$  on  $x$ . This can be written as the isometry condition  $V^\dagger V = I$  of the operator  $V : \psi \mapsto V(\cdot)\psi$  in terms of the integral

$$(2.1) \quad \int_y V(y)^\dagger V(y) d\mu_y = I, \quad \text{or} \quad \sum_y V(y)^\dagger V(y) = I.$$

with respect to the base measure  $\mu$  which is usually the counting measure,  $d\mu_y = 1$  in the discrete case, e.g. in the case of the projection-valued  $V(y) = F(y)$ . The general case of orthoprojectors  $V(y) = F(y)$  corresponds to the Krönicker  $\delta$ -function  $V(y) = \delta_y^X$  of a self-adjoint operator  $X$  on  $\mathfrak{h}$  with the discrete spectrum coinciding with the measured values  $y$ .

As in the simple example of the Schrödinger's cat the dynamical realization of such  $V$  can always be constructed in terms of a unitary transformation on an extended Hilbert space  $\mathfrak{h} \otimes \mathfrak{g}$  and a normalized wave function  $\chi^\circ \in \mathfrak{g}$ . It is easy to find such unitary dilation of any reduction family  $V$  of the form

$$(2.2) \quad V(y) = e^{-iE/\hbar} \exp \left[ -X \frac{d}{dy} \right] \varphi(y) = e^{-iE/\hbar} F(y),$$

given by a normalized wave-function  $\varphi \in L^2(\mathbb{G})$  on a cyclic group  $\mathbb{G} \ni y$  (e.g.  $\mathbb{G} = \mathbb{R}$  or  $\mathbb{G} = \mathbb{Z}$ ). Here the shift  $F(y) = \varphi(y - X)$  of  $\chi^\circ = \varphi$  by a measured operator  $X$  in  $\mathfrak{h}$  is well-defined by the unitary shifts  $\exp \left[ -x \frac{d}{dy} \right]$  in  $\mathfrak{g} = L^2(\mathbb{G})$  in the eigen-representation of any selfadjoint  $X$  having the spectral values  $x \in \mathbb{G}$ , and  $E = E^\dagger$  is any free evolution action after the measurement. As was noted by von Neumann for the case  $\mathbb{G} = \mathbb{R}$  in [11], the operator  $S = \exp \left[ -X \frac{d}{dy} \right]$  is unitary in  $\mathfrak{h} \otimes \mathfrak{g}$ , and it coincides on  $\psi \otimes \varphi$  with the isometry  $F = S(I \otimes \varphi)$  on each  $\psi \in \mathfrak{h}$  such that the unitary operator  $W = e^{-iE/\hbar} S$  dilates the isometry  $V = e^{-iE/\hbar} F$  in the sense

$$W(\psi \otimes \chi^\circ) = e^{-iE/\hbar} S(\psi \otimes \varphi) = e^{-iE/\hbar} F\psi, \quad \forall \psi \in \mathfrak{h}.$$

The wave function  $\chi^\circ = \varphi$  defines the initial probability distribution  $|\varphi(y)|^2$  of the pointer coordinate  $y$  which can be dispersionless only if  $\varphi$  is an eigen-function of the pointer operator  $Y = \hat{y}$  (multiplication operator by  $y$  in  $\mathfrak{g}$ ) corresponding to a discrete spectral value  $y^\circ$  as a predetermined initial value of the pointer,  $y^\circ = 0$  say. This corresponds to ortho-projectors  $V(y) = \delta_y^X = F(y)$  ( $E = 0$ ) indexed by  $y$  from a discrete cycle group,  $y \in \mathbb{Z}$  for the discrete  $X$  having eigenvalues  $x \in \mathbb{Z}$  say. Thus the projection postulate is always diluted by such shift operator  $S$  with  $\chi^\circ(y) = \delta_y^0$  given as the eigen-function  $\varphi(y) = \delta_y^0$  corresponding to the initial

value  $y = 0$  for the pointer operator  $Y = \hat{y}$  in  $\mathfrak{g} = L^2(\mathbb{Z})$  [15] (In the case of the Schrödinger's cat  $U$  was simply the shift  $W \pmod{2}$  in  $\mathfrak{g} = L^2(0, 1) := \mathbb{C}^2$ ).

There exist another, canonical construction of the unitary operator  $W$  with the eigen-vector  $\chi^\circ \in \mathfrak{g}$  for a 'pointer observable'  $Y$  in an extended Hilbert space  $\mathfrak{g}$  even if  $y$  is a continuous variable of the general family  $V(y)$ . More precisely, it can always be represented on the tensor product of the system space  $\mathfrak{h}$  and the space  $\mathfrak{g} = \mathbb{C} \oplus L_\mu^2$  of square-integrable functions  $\chi(y)$  defining also the values  $\chi(y^\circ) \in \mathbb{C}$  at an additional point  $y^\circ \neq y$  corresponding to the absence of a result  $y$  and  $\chi^\circ = 1 \oplus 0$  such that

$$(2.3) \quad \langle x | V(y) \psi = (\langle x | \otimes \langle y |) W (\psi \otimes \chi^\circ), \quad \forall \psi \in \mathfrak{h}$$

for each measured value  $y \neq y^\circ$ .

Now we prove this unitary dilation theorem for the general  $V(y)$  by the explicit construction of the matrix elements  $W_{y'}^y$  in the unitary block-operator  $W = [W_{y'}^y]$  defined as  $(I \otimes \langle y |) W (I \otimes |y'\rangle)$  by

$$\psi^\dagger W_{y'}^y \psi' = (\psi^\dagger \otimes \langle y |) W (\psi' \otimes |y'\rangle),$$

identifying  $y^\circ$  with 0 (assuming that  $y \neq 0$ , e.g.  $y = 1, \dots, n$ ). We shall use the short notation  $\mathfrak{f} = L_\mu^2$  for the functional Hilbert space on the measured values  $y$  and  $\chi^\circ = |y^\circ\rangle (=|0\rangle$  if  $y^\circ = 0$ ) for the additional state-vector  $\chi^\circ \in \mathfrak{g}$ , identifying the extended Hilbert space  $\mathfrak{g} = \mathbb{C} \oplus \mathfrak{f}$  with the space  $L_{\mu \oplus 1}^2$  of square-integrable functions of all  $y$  by the extension  $\mu \oplus 1$  of the measure  $\mu$  at  $y^\circ$  as  $d\mu_{y^\circ} = 1$ .

Indeed, we can always assume that  $V(y) = e^{-iE/\hbar} F(y)$  where the family  $F$  is viewed as an isometry  $F : \mathfrak{h} \rightarrow \mathfrak{h} \otimes \mathfrak{f}$  corresponding  $F^\dagger F = I$  (not necessarily of the form  $F(y) = \chi^\circ(y - X)$  as in (2.2)). Denoting  $e^{-iE/\hbar} F$  as the column of  $W_0^y$ ,  $y \neq 0$ , and  $e^{-iE/\hbar} F^\dagger$  as the row of  $W_y^0$ ,  $y \neq 0$ , we can compose the unitary block-matrix

$$(2.4) \quad [W_{y'}^y] := e^{-iE/\hbar} \begin{bmatrix} O & F^\dagger \\ F & I \otimes \hat{1} - FF^\dagger \end{bmatrix}, \quad I \otimes \hat{1} = [\text{Id}_{y'}^y]_{y' \neq 0}^{y \neq 0}$$

describing an operator  $W = [W_{y'}^y]$  on the product  $\mathfrak{h} \otimes \mathfrak{g}$ , where  $\mathfrak{g} = \mathbb{C} \oplus \mathfrak{f}$ , represented as  $\mathfrak{h} \oplus (\mathfrak{h} \otimes \mathfrak{f})$ ,  $\mathfrak{f} = L_\mu^2$ . It has the adjoint  $W^\dagger = e^{iE/\hbar} W e^{-iE/\hbar}$ , and obviously

$$(I \otimes \langle y |) W (I \otimes |0\rangle) = V(y), \quad \forall y \neq 0.$$

The unitarity  $W^{-1} = W^\dagger$  of the constructed operator  $W$  is the consequence of the isometricity  $F^\dagger F = I$  and thus the projectivity  $(FF^\dagger)^2 = FF^\dagger$  of  $FF^\dagger$  and of  $I \otimes \hat{1} - FF^\dagger$ :

$$W^\dagger W = \begin{bmatrix} F^\dagger F & F^\dagger (I \otimes \hat{1} - FF^\dagger) \\ (I \otimes \hat{1} - FF^\dagger) F & FF^\dagger + I \otimes \hat{1} - FF^\dagger \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \otimes \hat{1} \end{bmatrix}.$$

In general the observation may be incomplete: the data  $y$  may be the only observable part of a pair  $(z, y)$  defining the stochastic wave propagator  $V(z, y)$ . Consider for simplicity a discrete  $z$  such that

$$V^\dagger V := \sum_z \int V(z, y)^\dagger V(z, y) d\mu_y = I.$$

Then the linear unital map on the algebra  $\mathcal{B}(\mathfrak{h}) \otimes \mathcal{C}$  of the completely positive form

$$\pi(\hat{g}B) = \sum_z \int g(y) V(z, y)^\dagger B V(z, y) d\mu_y \equiv M[g\pi(B)]$$

describes the "Heisenberg picture" for generalized von Neumann reduction with an incomplete measurement results  $y$ . Here  $B \in \mathcal{B}(\mathfrak{h})$ ,  $\hat{g}$  is the multiplication operator by a measurable function of  $y$  defining any system-pointer observable by linear combinations of  $B(y) = g(y)B$ , and

$$\pi(y, B) = \sum_z V(z, y)^\dagger B V(z, y), \quad M[B(y)] = \int B(y) d\mu_y.$$

The function  $y \mapsto \pi(y)$  with values in the completely positive maps  $B \mapsto \pi(y, B)$ , or operations, is the basic tool in the operational approach to quantum measurements. Its adjoint

$$\pi^*(\sigma) = \sum_z V(y, z) \sigma V(y, z)^\dagger d\mu_y = \pi^*(y, \sigma) d\mu_y,$$

is given by the density matrix transformation and it is called the *instrument* in the phenomenological measurement theories. The operational approach was introduced by Ludwig [16], and the mathematical implementation of the notion of instrument was originated by Davies and Lewis [17].

An abstract instrument now is defined as the adjoint to a unital completely positive map  $\pi$  for which  $\pi_y^*(\sigma)$  is a trace-class operator for each  $y$ , normalized to a density operator  $\rho = \int d\pi_y^*(\sigma)$ . The quantum mixed state described by the operator  $\rho$  is called the *prior* state, i.e. the state which has been prepared for the measurement. A unitary dilation of the generalized reduction (or "instrumental") map  $\pi$  was constructed by Ozawa [18], but as we shall now see, this, as well as the canonical dilation (2.4), is only a preliminary step towards the its quantum stochastic realization allowing the dynamical derivation of the reduction postulate as a result of the statistical inference as it was suggested in [3].

### 3. THE FUTURE-PAST BOUNDARY VALUE PROBLEM

The additional system of the constructed unitary dilation for the measurement propagator  $V(y)$  represents only the pointer coordinate of the measurement apparatus  $y$  with the initial value  $y = y^\circ (= 0 \text{ corresponding to } \chi^\circ = |0\rangle)$ . It should be regarded as a classical system (like the Schrödinger's cat) at the instants of measurement  $t > 0$  in order to avoid the applying of the projection postulate for inferences in the auxiliary system. Indeed, the actual events of the measurement can be only those propositions  $E$  in the extended system which may serve as the conditions for any other proposition  $F$  as a potential in future event, otherwise there can't be any causality even in the weak, statistical sense. This means that future states should be statistically predictable in any prior state of the system in the result of testing the measurable event  $E$  by the usual conditional probability (Bayes) formula

$$(3.1) \quad \Pr\{F = 1|E = 1\} = \Pr\{E \wedge F = 1\} / \Pr\{E = 1\} \quad \forall F,$$

and this predictability, or statistical causality means that the prior quantum probability  $\Pr\{F\} \equiv \Pr\{F = 1\}$  must coincide with the statistical expectation of  $F$  as



the weighted sum

$$\Pr \{F|E\} \Pr \{E\} + \Pr \{F|E^\perp\} \Pr \{E^\perp\} = \Pr \{F\}$$

of this  $\Pr \{F|E\} \equiv \Pr \{F = 1|E = 1\}$  and the complementary conditional probability  $\Pr \{F|E^\perp\} = \Pr \{F = 1|E = 0\}$ . As one can easily see, this is possible if and only if (??) holds, i.e. any other future event-orthoprojector  $F$  of the extended system must be compatible with the actual event-orthoprojector  $E$ .

The actual events in the measurement model obtained by the unitary dilation are only the orthoprojectors  $E = I \otimes \hat{1}_\Delta$  on  $\mathfrak{h} \otimes \mathfrak{g}$  corresponding to the propositions " $y \in \Delta$ " where  $\hat{1}_\Delta$  is the multiplication by the indicator  $1_\Delta$  for a measurable on the pointer scale subset  $\Delta$ . Other orthoprojectors which are not compatible with these orthoprojectors, are simply not admissible as the questions by the choice of time arrow. This choice restores the quantum causality as statistical predictability, i.e. the statistical inference made upon the sample data. And the actual observables in question are only the measurable functions  $g(y)$  of  $y \neq y^\circ$  represented on  $\mathfrak{f} = L^2(\mu)$  by the commuting operators  $\hat{g}$  of multiplication by these functions,  $\langle y|\hat{g}\chi = g(y)\chi(y)$ . As follows from  $W_0^0 = O$ , the initial value  $y^\circ = 0$  is never observed at the time  $t = 1$ :

$$\|\psi_1(0)\|^2 = \|(I \otimes \langle 0|) W(\psi \otimes |0\rangle)\|^2 = \|W_0^0 \psi\|^2 = 0, \quad \forall \psi \in \mathfrak{h}$$

(that is a measurable value  $y \neq y^\circ$  is certainly observed at  $t = 1$ ). These are the only appropriate candidates for Bell's "beables", [1], p.174. Indeed, such commuting observables, extended to the quantum counterpart as  $G_0 = I \otimes \hat{g}$  on  $\mathfrak{h} \otimes \mathfrak{f}$ , are compatible with any admissible question or observable  $B$  on  $\mathfrak{h}$  represented with respect to the output states  $\psi_1 = W\psi_0$  at the time of measurement  $t = 1$  by an operator  $B_1 = B \otimes \hat{1}$  on  $\mathfrak{h} \otimes \mathfrak{f}$ . The probabilities (or, it is better to say, the propensities) of all such questions are the same in all states whether an observable  $G_0$  was measured but the result not read, or it was not measured at all. In this sense the measurement of  $G_0$  is called *nondemolition* with respect to the system observables  $B_1$ , they do not demolish the propensities, or prior expectations of  $B$ . However as we shall show now they are not necessary compatible with the same operators  $B$  of the quantum system at the initial stage and currently represented as  $WB_0W^\dagger$  on  $\psi_1$ , where  $B_0 = B \otimes \hat{1}$  is the Schrödinger representation of  $B$  at the time  $t = 0$  on the corresponding input states  $\psi_0 = W^\dagger\psi_1$  in  $\mathfrak{h} \otimes \mathfrak{g}$ .

Indeed, we can see this on the example of the Schrödinger cat, where  $W$  is the flip  $S$  in  $\mathfrak{g} = \mathbb{C}^2$  (shift mod 2). In this case the operators the operators  $\hat{g}_1$  in the Heisenberg picture  $G = S^\dagger G_0 S$  are represented on  $\mathfrak{h} \otimes \mathfrak{g}$  as the diagonal operators  $G = [g(\tau + v) \delta_\tau^\tau, \delta_v^v]$  of multiplication by  $g(\tau + v)$ , where the sum  $\tau + v = |\tau - v|$  is modulo 2. Obviously they do not commute with  $B_0$  unless  $B$  is also a diagonal operator  $\hat{f}$  of multiplication by a function  $f(\tau)$ , in which case

$$[B_0, G] \psi_0(\tau, v) = [f(\tau), g(\tau + v)] \psi_0(\tau, v) = 0, \quad \forall \psi_0 \in \mathfrak{h} \otimes \mathfrak{g}.$$

The restriction of the possibilities in a quantum system to only the diagonal operators  $B = \hat{f}$  of the atom which would eliminate the time arrow in the nondemolition condition, amounts to the redundancy of the quantum consideration: all such (possible and actual) observables can be simultaneously represented as classical observables by the measurable functions of  $(\tau, v)$ .

Thus the constructed semiclassical algebra  $\mathcal{B}_- = \mathcal{B}(\mathfrak{h}) \otimes \mathcal{C}$  of the Schrödinger's atom and the pointer (dead or alive cat) is not dynamically invariant in the sense

that transformed algebra  $W^\dagger \mathcal{B}_- W$  does not coincide and is not a part of  $\mathcal{B}_-$  but of  $\mathcal{B}_+ = \mathcal{B}(\mathfrak{h}) \otimes \mathcal{B}(\mathfrak{g})$ . This is also true in the general case, unless all the system-pointer observables in the Heisenberg picture are still decomposable,

$$W^\dagger (B \otimes \hat{g}) W = \int |y\rangle g(y) B(y) \langle y| d\mu_y,$$

which would imply  $W^\dagger \mathcal{B} W \subseteq \mathcal{B}$ . (Such dynamical invariance of the decomposable algebra, given by the operator-valued functions  $B(y)$ , can be achieved by this unitary dilations only in trivial cases.) This is why the von Neumann type dilation (??), and even more general dilations (2.4), or [18, 3] cannot yet be considered as the dynamical solution of the instantaneous quantum measurement problem which we formulate in the following way.

*Given a reduction postulate defined by an isometry  $V$  on  $\mathfrak{h}$  into  $\mathfrak{h} \otimes \mathfrak{g}$ , find a triple  $(\mathcal{G}, \mathfrak{A}, \Phi^\circ)$  consisting of Hilbert space  $\mathcal{G} = \mathcal{G}_- \otimes \mathcal{G}_+$  embedding the Hilbert spaces  $\mathfrak{f} = L_\mu^2$  by an isometry into  $\mathcal{G}_+$ , an algebra  $\mathfrak{A} = \mathfrak{A}_- \otimes \mathfrak{A}_+$  on  $\mathcal{G}$  with an Abelian subalgebra  $\mathfrak{A}_- = \mathfrak{C}$  generated by an observable (beable)  $Y$  on  $\mathcal{G}_-$ , and a state-vector  $\Phi^\circ = \Phi_-^\circ \otimes \Phi_+^\circ \in \mathcal{G}$  such that there exist a unitary operator  $U$  on  $\mathcal{H} = \mathfrak{h} \otimes \mathcal{G}$  which induces an endomorphism on the product algebra  $\mathfrak{B} = \mathcal{B}(\mathfrak{h}) \otimes \mathfrak{A}$  in the sense  $U^\dagger A U \in \mathfrak{B}$  for all  $B \in \mathfrak{B}$ , with*

$$\pi(\hat{g} \otimes B) := (I \otimes \Phi^\circ)^\dagger U^\dagger (B \otimes g(Y)) U (I \otimes \Phi^\circ) = M[gV^\dagger B V]$$

for all  $B \in \mathcal{B}(\mathfrak{h})$  and measurable functions  $g$  of  $Y$ , where  $M[B] = \int B(y) \mu_y$ .

It is always possible to achieve this dynamical invariance by extending the classical measurement apparatus' to an infinite auxiliary semi-classical system. Here we sketch this construction for the general unitary dilation (2.4).

The construction consists of five steps. The first, preliminary step of a unitary dilation for the isometry  $V$  has been already described in the previous Section.

Second, we construct the triple  $(\mathcal{G}, \mathfrak{A}, \Phi^\circ)$ . Denote by  $\mathfrak{g}_s$ ,  $s = \pm 0, \pm 1, \dots$  (the indices  $\pm 0$  are distinct and ordered as  $-0 < +0$ ) the copies of the Hilbert space  $\mathfrak{g} = \mathbb{C} \oplus \mathfrak{f}$  in the dilation (2.4) represented as the functional space  $L_\mu^2$  on the values of  $y$  including  $y^\circ = 0$ , and  $\mathbb{G}_n = \mathfrak{g}_{-n} \otimes \mathfrak{g}_{+n}$ ,  $n \geq 0$ . We define the Hilbert space of the past  $\mathcal{G}_-$  and the future  $\mathcal{G}_+$  as the state-vector spaces of semifinite discrete strings generated by the infinite tensor products  $\Phi_- = \chi_{-0} \otimes \chi_{-1} \otimes \dots$  and  $\Phi_+ = \chi_{+0} \otimes \chi_{+1} \otimes \dots$  with all but finite number of  $\chi_s \in \mathfrak{g}_s$  equal to the initial state  $\chi_s^\circ$ , the copies of  $\chi^\circ = |0\rangle \in \mathfrak{g}$ . Denoting by  $\mathcal{A}_s$  the copies of the algebra  $\mathcal{B}(\mathfrak{g})$  of bounded operators if  $s \geq +0$ , of the diagonal subalgebra  $\mathcal{D}(\mathfrak{g})$  on  $\mathfrak{g}$  if  $s \leq -0$ , and  $\mathcal{A}_n = \mathcal{A}_{-n} \otimes \mathcal{A}_{+n}$  we construct the algebras of the past  $\mathfrak{A}_-$  and the future  $\mathfrak{A}_+$  and the whole algebra  $\mathfrak{A}$ .  $\mathfrak{A}_\pm$  are generated on  $\mathcal{G}_\pm$  respectively by the diagonal operators  $\hat{f}_{-0} \otimes \hat{f}_{-1} \otimes \dots$  and by  $X_{+0} \otimes X_{+1} \otimes \dots$  with all but finite number of  $\hat{f}_s \in \mathcal{A}_s$ ,  $s < 0$  and  $X_s \in \mathcal{A}_s$ ,  $s > 0$  equal the identity operator  $\hat{1}$  in  $\mathfrak{g}$ . Here  $\hat{f}$  stands for the multiplication operator by a function  $f$  of  $y \in \mathbb{R}$ , in particular,  $\hat{y}$  is the multiplication by  $y$ , with the eigen-vector  $\chi^\circ = |0\rangle$  corresponding to the eigen-value  $y^\circ = 0$ . The Hilbert space  $\mathcal{G}_- \otimes \mathcal{G}_+$  identified with  $\mathcal{G} = \otimes \mathbb{G}_n$ , the decomposable algebra  $\mathfrak{A}_- \otimes \mathfrak{A}_+$  identified with  $\mathfrak{A} = \otimes \mathcal{A}_n$  and the product vector  $\Phi_- \otimes \Phi_+$  identified with  $\Phi = \otimes \phi_n \in \mathcal{G}$ , where  $\phi_n = \chi_{-n} \otimes \chi_{+n} \equiv \chi_{-n} \chi_{+n}$  with all  $\chi_s = \chi^\circ$  stand as candidates for the triple  $(\mathcal{G}, \mathfrak{A}, \Phi)$ . Note that the eigen-vector  $\Phi^\circ = \otimes \phi_n^\circ$  with all  $\phi_n^\circ = \chi^\circ \otimes \chi^\circ$  corresponds to the initial eigen-state  $y^\circ = 0$  of all observables  $Y_{\pm n} = \hat{1}_0 \otimes \dots \otimes \hat{1}_{n-1} \otimes \hat{y}_\pm \otimes \hat{1}_{n+1} \otimes \dots$  in  $\mathcal{G}$ , where  $\hat{1} = \hat{1}_- \otimes \hat{1}_+$ ,  $\hat{y}_- = \hat{y} \otimes \hat{1}_+$ ,  $\hat{y}_+ = \hat{1}_- \otimes \hat{y}$  and  $\hat{1}_{\pm n}$  are the identity operators in  $\mathfrak{g}_{\pm n}$ .

Third, we define the unitary evolution on the product space  $\mathfrak{h} \otimes \mathcal{G}$  of the total system by

$$(3.2) \quad U : \psi \otimes \chi_- \chi_+ \otimes \chi_{-1} \chi_{+1} \cdots \mapsto W(\psi \otimes \chi_+) \chi_{+1} \otimes \chi_- \chi_{+2} \cdots,$$

incorporating the right shift in  $\mathcal{G}_-$ , the left shift in  $\mathcal{G}_+$  and the conservative boundary condition  $W : \mathfrak{h} \otimes \mathfrak{g}_+ \rightarrow \mathfrak{h} \otimes \mathfrak{g}_-$  given by the unitary dilation (2.4). We have obviously

$$(I \otimes \langle y_-, y_+, y_{-1}, y_{+1} \cdots |) U (I \otimes |y_-^0, 0, y_{-1}^0, 0 \cdots \rangle) = \cdots \delta_0^{y_{+1}} \delta_0^{y_+} V(y_-) \delta_{y_-^0}^{y_{-1}} \delta_{y_{-1}^0}^{y_0} \cdots$$

so that the extended unitary operator  $U$  still reproduces the reduction  $V(y)$  in the result  $y \neq y^\circ$  of the measurement  $Y = Y_{-0}$  in a sequence  $(Y_{-0}, Y_{+0}, Y_{-1} Y_{+1}, \dots)$  with all other  $y_s$  being zero  $y^\circ = 0$  with the probability one for the initial ground state  $\Phi^\circ$  of the connected string.

Fourth, we prove the dynamical invariance  $U^\dagger (\mathcal{B}(\mathfrak{h}) \otimes \mathfrak{A}) U \subseteq \mathcal{B}(\mathfrak{h}) \otimes \mathfrak{A}$  of the decomposable algebra of the total system, incorporating the measured quantum system  $\mathcal{B}(\mathfrak{h})$  as the boundary between the quantum future (the right string considered as quantum,  $\mathfrak{A}_+ = \mathcal{B}(\mathcal{G}_+)$ ) with the classical past (the left string considered as classical,  $\mathfrak{A}_- = \mathcal{D}(\mathcal{G}_-)$ ). This follows straightforward from the definition of  $U$

$$U^\dagger (B \otimes \hat{g}_- X_+ \otimes \hat{g}_{-1} X_{+1} \cdots) U = \hat{g}_{-1} W^\dagger (B \otimes \hat{g}_-) W \otimes \hat{g}_{-2} X_+ \cdots$$

due to  $W^\dagger (B \otimes \hat{g}) W \in \mathcal{B}(\mathfrak{h}) \otimes \mathcal{B}(\mathfrak{g})$  for all  $\hat{g} \in \mathcal{D}(\mathfrak{g})$ . However this algebra representing the total algebra  $\mathcal{B}(\mathfrak{h}) \otimes \mathfrak{A}$  on  $\mathfrak{h} \otimes \mathcal{G}$  is not invariant under the inverse transformation, and there is no way to achieve the inverse invariance keeping  $\mathfrak{A}$  decomposable as the requirement for statistical causality of quantum measurement if  $W(B \otimes X) W^\dagger \notin \mathcal{B}(\mathfrak{h}) \otimes \mathcal{D}(\mathfrak{g})$  for some  $B \in \mathcal{B}(\mathfrak{h})$  and  $X \in \mathcal{B}(\mathfrak{g})$ :

$$U(B \otimes \hat{g}_- X_+ \otimes \hat{g}_{-1} X_{+1} \cdots) U^\dagger = W(B \otimes X_+) W^\dagger X_{+1} \otimes \hat{g}_- X_{+2} \cdots.$$

And the fifth step is to explain on this dynamical model the decoherence phenomenon, irreversibility and causality by giving a constructive scheme in terms of equation for quantum predictions as statistical inferences by virtue of gaining the measurement information.

Because of the crucial importance of these realizations for developing understanding of the mathematical structure and interpretation of modern quantum theory, we need to analyze the mathematical consequences which can be drawn from such schemes.

#### 4. DECOHERENCE AND QUANTUM PREDICTION

The analysis above shows that the dynamical realization of a quantum instantaneous measurement is possible in an infinitely extended system, but the discrete unitary group of unitary transformations  $U^t$ ,  $t \in \mathbb{N}$  with  $U^1 = U$  induces not a group of Heisenberg automorphisms but an injective irreversible semigroup of endomorphisms on the decomposable algebra  $\mathfrak{B} = \mathcal{B}(\mathfrak{h}) \otimes \mathfrak{A}$  of this system. However it is locally invertible on the center of the algebra  $\mathfrak{A}$  in the sense that it reverses the shift dynamics on  $\mathfrak{A}^{[0]}$ :

$$(4.1) \quad T_{-t} (I \otimes Y_s) T_t := I \otimes Y_{s-t} = U^t (I \otimes Y_s) U^{-t}, \quad \forall s \leq -0, t \in \mathbb{N}.$$

Here  $Y_{-n} = \hat{1}^{\otimes n} \otimes \hat{g}_- \otimes I_n$ , where  $I_n = \otimes_{k>n} \hat{1}_k$ , and  $T_{-t} = (T)^t$  is the power of the isometric shift  $T : \Phi_- \mapsto \chi^\circ \otimes \Phi_-$  on  $\mathcal{G}_-$  extended to the free unitary dynamics

of the whole system as

$$T : \psi \otimes \chi_- \chi_+ \otimes \chi_{-1} \chi_{+1} \cdots \mapsto \psi \otimes \chi_+ \chi_{+1} \otimes \chi_- \chi_{+2} \cdots$$

The extended algebra  $\mathfrak{B}$  is the minimal algebra containing all consistent events of the history and all admissible questions about the future of the open system under observation initially described by  $\mathcal{B}(\mathfrak{h})$ . Indeed, it contains all Heisenberg operators

$$B(t) = U^{-t}(B \otimes I)U^t, \quad Y_-(t) = U^{-t}(I \otimes Y_{-0})U^t, \quad \forall t > 0$$

of  $B \in \mathcal{B}(\mathfrak{h})$ , and these operators not only commute at each  $t$ , but also satisfy the *nondemolition causality condition*

$$(4.2) \quad [B(t), Y_-(r)] = 0, \quad [Y_-(t), Y_-(r)] = 0, \quad \forall t \geq r \geq 0.$$

This follows from the commutativity of the Heisenberg string operators

$$Y_{r-t}(t) = U^{-t}(I \otimes Y_{r-t})U^t = Y_-(r)$$

at the different points  $s = r - t < 0$  coinciding with  $Y_s(r - s)$  for any  $s < 0$  because of (4.1), and also from the commutativity with  $B(t)$  due to the simultaneous commutativity of all  $Y_s(0) = I \otimes Y_s$  and  $B(0) = B \otimes I$ . Thus all output Heisenberg operators  $Y_-(r)$ ,  $0 < r \leq t$  at the boundary of the string can be measured simultaneously as  $Y_{-n}(t) = Y_-(t - n)$  at the different points  $n < t$ , or sequentially at the point  $s = -0$  as the commutative nondemolition family  $Y_0^{[t]} = (Y^1, \dots, Y^t)$ , where  $Y^r = Y_-(r)$ . This defines the reduced evolution operators

$$V(t, y_0^{[t]}) = V(y^t) V(y^{t-1}) \cdots V(y^1), \quad t > 0$$

of a *sequential measurement* in the system Hilbert space  $\mathfrak{h}$  with measurement data  $y_0^{[t]} = \{(0, t] \ni r \mapsto y^r\}$ . One can prove this using the *filtering recurrency equation*

$$(4.3) \quad \psi(t, y_0^{[t]}) = V(y_t) \psi(t-1, y_0^{[t-1]}), \quad \psi(0) = \psi$$

for  $\psi(t, y_0^{[t]}) = V(t, y_0^{[t]}) \psi$  and for  $\Psi(t) = U^t(\psi \otimes \Phi_- \otimes \Phi_+^\circ)$ , where  $\psi \in \mathfrak{h}$ , and  $V(y_t)$  is defined by

$$(I \otimes \langle y_{-\infty}^{[t]} | \otimes \langle y_t^\infty |) U \Psi(t-1) = V(y_t) \psi(t-1, y_0^{[t-1]}) \langle \delta_0^{y_t^\infty} y_{-\infty}^{[0]} | \Phi_-.$$

Moreover, any future expectations in the system, say the probabilities of the questions  $F(t) = U^{-t}(F \otimes I)U^t$ ,  $t \geq s$  given by orthoprojectors  $F$  on  $\mathfrak{h}$ , can be statistically predicted upon the results of the past measurements of  $Y_-(r)$ ,  $0 < r \leq t$  and initial state  $\psi$  by the simple conditioning

$$\Pr\{F(t) | E(dy^1 \times \cdots \times dy^t)\} = \frac{\Pr\{F(t) \wedge E(dy^1 \times \cdots \times dy^t)\}}{\Pr\{E(dy^1 \times \cdots \times dy^t)\}}.$$

Here  $E$  is the joint spectral measure for  $Y^1, \dots, Y^t$ , and the probabilities in the numerator (and denominator) are defined as

$$\|F(t) E(dy^1 \times \cdots \times dy^t) (\psi \otimes \Phi^\circ)\|^2 = \|F \psi(t, y_0^{[t]})\|^2 d\mu_{y^1} \cdots d\mu_{y^t}$$

(and for  $F = I$ ) due to the commutativity of  $F(t)$  with  $E_-$ . This implies the usual sequential instrumental formula

$$\langle B \rangle(t, y_0^r) = \frac{\psi^\dagger \pi(t, y_0^r, B) \psi}{\psi^\dagger \pi(t, y_0^r, I) \psi} = M \left[ \psi_{y_0^r}^\dagger(t) B \psi_{y_0^r}(t) | y_0^r \right]$$

for the future expectations of  $B(t)$  conditioned by  $Y_-(1) = y^1, \dots, Y_-(s) = y^s$  for any  $t > r$ . Here  $\psi_{y_0^r}(t) = \psi(t, y_0^r) / \|\psi(t, y_0^r)\|$ , and

$$\pi(t, y_0^r, B) = \int \cdots \int V(t, y_0^r, y_r^t)^\dagger B V(t, y_0^r, y_r^t) d\mu_{y_{r+1}} \cdots d\mu_{y_t}$$

is the sequential reduction map  $V(t, y_0^t)^\dagger B V(t, y_0^t)$  defining the prior probability distribution

$$P(dy_0^t) = \psi^\dagger \pi(t, y_0^t, I) \psi d\mu_{y_0^t} = \|\psi(t, y_0^t)\|^2 d\mu_{y^1} \cdots d\mu_{y_t}$$

integrated over  $y_r^t$  if these data are ignored for the quantum prediction of the state at the time  $t > r$ .

Note that the stochastic vector  $\psi(t, y_0^t)$ , normalized as

$$\int \cdots \int \|\psi(t, y_0^t)\|^2 d\mu_{y^1} \cdots d\mu_{y_t} = 1$$

depends linearly on the initial state vector  $\psi \in \mathfrak{h}$ . However *the posterior state vector*  $\psi_{y_0^t}(t)$  is nonlinear, satisfying the *nonlinear stochastic recurrency equation*

$$(4.4) \quad \psi_{y_0^t}(t) = V_{y_0^{t-1}}(t, y^t) \psi_{y_0^{t-1}}(t-1), \quad \psi(0) = \psi,$$

where  $V_{y_0^{t-1}}(t, y^t) = \|V(t-1, y_0^{t-1})\psi\| V(y^{t-1}) / \|V(t, y_0^t)\psi\|$ .

In particular one can always realize in this way any sequential observation of the noncommuting operators  $B_t = e^{iE/\hbar} B_0 e^{-iE/\hbar}$  given by a selfadjoint operator  $B_0$  with discrete spectrum and the energy operator  $E$  in  $\mathfrak{h}$ . It corresponds to the sequential collapse given by  $V(y) = \delta_0^{B_0} e^{-iE/\hbar}$ . Our construction suggests that any demolition sequential measurement can be realized as the nondemolition by the commutative family  $Y_-(t)$ ,  $t > 0$  with a common eigenvector  $\Phi^\circ$  as the pointers initial state, satisfying the causality condition (4.2) with respect to all future Heisenberg operators  $B(t)$ . And the sequential collapse (4.4) follows from the usual Bayes formula for conditioning of the compatible observables due to the classical inference in the extended system. Thus, we have solved the *sequential quantum measurement problem* which can rigorously be formulated as

*Given a sequential reduction family  $V(t, y_0^t)$ ,  $t \in \mathbb{N}$  of isometries resolving the filtering equation (4.3) on  $\mathfrak{h}$  into  $\mathfrak{h} \otimes \mathfrak{f}^{\otimes t}$ , find a triple  $(\mathcal{G}, \mathfrak{A}, \Phi)$  consisting of a Hilbert space  $\mathcal{G} = \mathcal{G}_- \otimes \mathcal{G}_+$  embedding all tensor products  $\mathfrak{f}^{\otimes t}$  of the Hilbert spaces  $\mathfrak{f} = L_\mu^2$  by an isometry into  $\mathcal{G}_+$ , an algebra  $\mathfrak{A} = \mathfrak{A}_- \otimes \mathfrak{A}_+$  on  $\mathcal{G}$  with an Abelian subalgebra  $\mathfrak{A}_- = \mathfrak{C}$  generated by a compatible discrete family  $Y_{-\infty}^0 = \{Y_s, s \leq 0\}$  of the observables (beables)  $Y_s$  on  $\mathcal{G}_-$ , and a state-vector  $\Phi^\circ = \Phi_-^\circ \otimes \Phi_+^\circ \in \mathcal{G}$  such that there exist a unitary group  $U^t$  on  $\mathcal{H} = \mathfrak{h} \otimes \mathcal{G}$  inducing a semigroup of endomorphisms*

$\mathfrak{B} \ni B \mapsto U^{-t} B U^t \in \mathfrak{B}$  on the product algebra  $\mathfrak{B} = \mathcal{B}(\mathfrak{h}) \otimes \mathfrak{A}$  (4.1 on  $\mathfrak{A}$ , with

$$\pi^t(\hat{g}_{-t} \otimes B) = (I \otimes \Phi^\circ)^\dagger U^{-t} \left( g_{-t} \left( Y_{-t}^{[0]} \right) \otimes B \right) U^t (I \otimes \Phi^\circ) = M \left[ gV(t)^\dagger B V(t) \right]$$

for any  $B \in \mathcal{B}(\mathfrak{h})$  and any operator  $\hat{g}_{-t} = \hat{g}_{-t} \left( Y_{-t}^{[0]} \right) \in \mathfrak{C}$  represented as the shifted function  $\hat{g}_{-t} \left( y_{-t}^{[0]} \right) = g \left( y_0^{[t]} \right)$  of  $Y_{-t}^{[0]} = (Y_{1-t}, \dots, Y_0)$  on  $\mathcal{G}$  by any measurable function  $g$  of  $y_0^{[t]} = (y_1, \dots, y_t)$  with arbitrary  $t > 0$ , where

$$M \left[ gV(t)^\dagger B V(t) \right] = \int \cdots \int g \left( y_0^{[t]} \right) V \left( t, y_0^{[t]} \right)^\dagger B V \left( t, y_0^{[t]} \right) d\mu_{y_1} \cdots d\mu_{y_t}.$$

Note that our construction of the solution to this problem admits also the time reversed representation of the sequential measurement process described by the isometry  $V$ . The reversed system leaves in the same Hilbert space, with the same initial state-vector  $\Phi^\circ$  in the auxiliary space  $\mathcal{G}$ , however the reversed auxiliary system is described by the reflected algebra  $\tilde{\mathfrak{A}} = R\mathfrak{A}R$  where the reflection  $R$  is described by the unitary flip-operator  $R : \Phi_- \otimes \Phi_+ \mapsto \Phi_+ \otimes \Phi_-$  on  $\mathcal{G} = \mathcal{G}_- \otimes \mathcal{G}_+$ . The past and future in the reflected algebra  $\tilde{\mathfrak{A}} = \mathfrak{A}_+ \otimes \mathfrak{A}_-$  are flipped such that its left subalgebra consists now of all operators on  $\mathcal{G}_-$ ,  $\tilde{\mathfrak{A}}_- = \mathcal{B}(\mathcal{G}) \supset \mathfrak{A}_-$  and its right subalgebra is the diagonal algebra  $\tilde{\mathfrak{A}}_+ = \mathcal{D}(\mathcal{G}_+) \subset \mathfrak{A}_+$  on  $\mathcal{G}_+$ . The inverse operators  $U^t, t < 0$  induce the reversed dynamical semigroup of the injective endomorphisms  $B \mapsto U^{-t} B U^t$  which leaves invariant the algebra  $\tilde{\mathfrak{B}} = \mathcal{B}(\mathfrak{h}) \otimes \tilde{\mathfrak{A}}$  but not  $\mathfrak{B}$ . The reversed canonical measurement process is described by another family  $Y_{[+0]}^\infty = (Y_{+t})$  of commuting operators  $Y_{+t} = RY_{-t}R$  in  $\tilde{\mathfrak{A}}_+$ , and the Heisenberg operators

$$Y_+(t) = Y_s(t-s) = RY_-(-t)R, \quad t < 0, s > 0,$$

are compatible and satisfy the reversed causality condition

$$[B(t), Y_+(r)] = 0, \quad [Y_+(t), Y_+(r)] = 0, \quad \forall t \leq r \leq 0.$$

It reproduces another, reversed sequence of the successive measurements

$$V^* \left( t, y_{[t]}^0 \right) = V^*(y_t) V^*(y_{t+1}) \cdots V^*(y_{-1}), \quad t < 0,$$

where  $V^*(y) = (I \otimes \langle y |) W^{-1} (I \otimes |0\rangle)$  depends on the choice of the unitary dilation  $W$  of  $V$ . In the case of the canonical dilation (2.4) uniquely defined up to the system evolution between the measurements, we obtain  $V^*(y) = F(y) e^{iE/\hbar}$ . If the system the Hamiltonian is time-symmetric, i.e.  $\bar{E} = E$  in the sense  $E\bar{\psi} = \overline{E\psi}$  with respect to the complex (or another) conjugation in  $\mathfrak{h}$ , and if  $\bar{F}(y) = e^{-iE/\hbar} F(\tilde{y}) e^{iE/\hbar}$ , where  $y \mapsto \tilde{y}$  is a covariant flip,  $\tilde{\tilde{y}} = y$  (e.g.  $\tilde{y} = y$ , or reflection of the measurement data under the time reflection  $t \mapsto -t$ ), then  $V^*(y) = \bar{V}(\tilde{y})$ . This means that the reversed measurement process can be described as time-reflected direct measurement process under the  $*$ -conjugation  $\psi^*(y) = \bar{\psi}(\tilde{y})$  in the space  $\mathfrak{h} \otimes \mathfrak{f}$ . And it can be modelled as the time reflected direct nondemolition process under the involution  $J(\psi \otimes \Phi) = \bar{\psi} \otimes R\Phi^*$  induced by  $\chi^*(y) = \bar{\chi}(\tilde{y})$  in  $\mathfrak{g}$  with the flip-invariant eigen-value  $y^\circ = 0$  and  $|0\rangle^* = |0\rangle$  corresponding to the real ground state  $\chi^\circ(y) = \delta_y^0$ .

Thus, the choice of time arrow, which is absolutely necessary for restoring statistical causality in quantum theory, is equivalent to a superselection rule. This corresponds to a choice of the minimal algebra  $\mathfrak{B} \subset \mathcal{B}(\mathcal{H})$  generated by all admissible questions on a suitable Hilbert space  $\mathcal{H}$  of the nondemolition representation

for a process of the successive measurements. All consistent events should be drawn from the center of  $\mathfrak{B}$ : the events must be compatible with the questions, otherwise the propensities for the future cannot be inferred from the testing of the past. The decoherence is dynamically induced by a unitary evolution from any pure state on the algebra  $\mathfrak{B}$  corresponding to the initial eigen-state for the measurement apparatus pointer which is described by the center of  $\mathfrak{B}$ . Moreover, the reversion of the time arrow corresponds to another choice of the admissible algebra. It can be implemented by a complex conjugation  $J$  on  $\mathcal{H}$  on the transposed algebra  $\tilde{\mathfrak{B}} = J\mathfrak{B}J$ . Note that the direct and reversed dynamics respectively on  $\mathfrak{B}$  and on  $\tilde{\mathfrak{B}}$  are only endomorphic, and that the invertible automorphic dynamics induced on the total algebra  $\mathcal{B}(\mathcal{H}) = \mathfrak{B} \vee \tilde{\mathfrak{B}}$  does not reproduce the decoherence due to the redundancy of one of its part for a given time arrow  $t$ .

The constructed dynamical realization of the instantaneous and sequential measurements is the simple discrete-time analog of the solution to the continuous boundary-value problem for quantum stochastic models of the nondemolition measurements. This boundary value problem, which was obtained recently by second quantization of the Dirac-type boundary value problem [19] for wave propagation on  $\mathbb{R}_+$ , gives an implementation of an old idea of Schrödinger [20] that the quantum jumps and measurements should be derived from a boundary value problem for “waves from future” interacting with the opposite “messages from the past”. This also gives a simple exactly solvable model in line with more recent attempts of the transactional interpretation of quantum mechanics [21]. The superselection causality principle which enables such purely dynamical interpretation for quantum measurements allows only the present and future to be quantum, defining the past as classical, stored in the trajectories of the particles. As Lawrence Bragg, a Nobel prize winner, once said, everything in the future is a wave, everything in the past is a particle.

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